Simple and efficient representations for the fundamental solutions of Stokes flow in a half-space

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Abstract

We derive new formulas for the fundamental solutions of slow, viscous flow, governed by the Stokes equations, in a half-space. They are simpler than the classical representations obtained by Blake and collaborators, and can be efficiently implemented using existing fast solvers libraries. We show, for example, that the velocity field induced by a Stokeslet can be annihilated on the boundary (to establish a zero slip condition) using a single reflected Stokeslet combined with a single Papkovich-Neuber potential that involves only a scalar harmonic function. The new representation has a physically intuitive interpretation.

1 Introduction

Viscous flow of passive and active suspensions in the presence of an infinite planar boundary is an important physical model in many areas of science and engineering. It serves as a useful paradigm for understanding the effect of confined geometries on the macroscopic flow behavior of particulate flows, for example, that of bacterial propulsion, cellular blood flow and colloidal suspensions, [15, 7, 3, 23, 21]. In problems where the Reynolds number is low, the ambient fluid is governed by the Stokes equations:

$$\mu\, \Delta \mathbf{u}(\mathbf{x}) = \nabla p(\mathbf{x}), \quad \nabla \cdot \mathbf{u}(\mathbf{x}) = 0, \tag{1}$$

where μ is the fluid viscosity, $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x}))$ is the velocity of the fluid, and $p(\mathbf{x})$ is the pressure. Assuming the plane wall is located at $x_3 = 0$ and that the flow velocity decays in the far field,

$$\mathbf{u}(\mathbf{x}) \to 0, \quad \text{as} \quad |\mathbf{x}| \to \infty,$$

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the no-slip boundary condition is

$$\mathbf{u}\big|_{x_3=0} = 0. \tag{2}$$

Boundary integral methods are particularly well-suited for problems of this kind since they discretize the domain boundary alone (resulting in many fewer degrees of freedom) and impose the decay and the no-slip conditions exactly. Moreover, they avoid the need for artificial truncation of the computational domain and can be solved rapidly and with great accuracy using fast algorithms such as the fast multipole method (FMM) and high order accurate quadrature rules.

In order to reformulate the Stokes equations as a boundary integral equation, however, one needs to have access to the Green's functions for the half-space [16, 14]. A now classical approach to constructing this Green's function is due to Blake and others [4, 5, 6, 1, 2, 20, 29]. Unfortunately, the resulting formulas are rather complicated, making them somewhat difficult to implement.

Here, we show that a much simpler alternative to the Blake solution can be obtained by combining a free space image, which annihilates the *tangential* components of velocity, with a Papkovich-Neuber potential [19, 18] which annihilates the normal component.

The present paper is organized as follows. We discuss Papkovich-Neuber potentials and the standard fundamental solutions for the Stokes equations in section 2. (See, for example, [16, 14, 15, 20]). We also review Blake's formula for the Stokeslet in a half-space. In section 3, we derive the image structures for Stokeslets, stresslets, rotlets, and Stokes doublets.

In the appendices, we provide the analogous formulas for two-dimensional half-space Stokes kernels and for some problems of linear elasticity.

2 Fundamental solutions, the Papkovich-Neuber representation, and Blake's formulas

Before discussing the various standard fundamental solutions for the Stokes equations in free-space, we introduce the Papkovich-Neuber representation, originally developed in [19, 18] for problems of linear elasticity. Without loss of generality, we assume that the fluid viscosity $\mu = 1$ in the rest of the paper.

Definition 2.1. *Let $\phi(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^3$ be a harmonic function. Then the induced Papkovich-Neuber representation is defined to be the paired vector field $\mathbf{u}$ and scalar field p given by:*

$$\mathbf{u}(\boldsymbol{x}) = x_3 \nabla \phi(\boldsymbol{x}) - \begin{bmatrix} 0 \\ 0 \\ \phi(\boldsymbol{x}) \end{bmatrix}, \quad p(\boldsymbol{x}) = 2\frac{\partial \phi(\boldsymbol{x})}{\partial x_3}. \tag{3}$$

It is straightforward to verify that $(\mathbf{u}, p)$ satisfy the Stokes equations (1) (with $\mu = 1$).

Suppose now that a force vector $\mathbf{f} = (f_1, f_2, f_3)$ is applied to a viscous fluid at a point $\mathbf{y}$. Then, it is well-known that the induced velocity and pressure can be computed using the Stokeslet (the single layer kernel):

$$S_{ij}(\mathbf{x}, \mathbf{y}) = \frac{1}{8\pi}\left[\frac{\delta_{ij}}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x}-\mathbf{y}|^3}\right], \quad i, j = 1, 2, 3, \tag{4}$$

$$P_j(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi}\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^3}, \tag{5}$$

where δ_{ij} is the Kronecker delta. More precisely, the velocity vector $\mathbf{u}(\mathbf{x})$ and pressure $p(\mathbf{x})$ are given by

$$u_i(\mathbf{x}) = \sum_{j=1}^{3} S_{ij}(\mathbf{x},\mathbf{y}) f_j, \quad p(\mathbf{x}) = \sum_{j=1}^{3} P_j(\mathbf{x},\mathbf{y}) f_j. \tag{6}$$

The stresslet (or double layer kernel) for the Stokes equations describes the velocity induced by an infinitesimal displacement $\mathbf{g} = (g_1, g_2, g_3)$ (sometimes called the *double force* source strength) at a point $\mathbf{y}$ with orientation vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$. It is given by:

$$T_{ijk}(\mathbf{x},\mathbf{y}) = \frac{3}{4\pi} \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^5}, \quad i,j,k = 1,2,3, \tag{7}$$

$$\Pi_{jk}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \left[-\frac{\delta_{jk}}{|\mathbf{x}-\mathbf{y}|^3} + \frac{3(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^5} \right], \tag{8}$$

and the corresponding formulas for the velocity and pressure at an arbitrary point $\mathbf{x}$ are

$$u_i(\mathbf{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}(\mathbf{x},\mathbf{y}) \nu_k g_j, \quad p(\mathbf{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} \Pi_{jk}(\mathbf{x},\mathbf{y}) \nu_k g_j. \tag{9}$$

2.1 Stokes flow in a half-space, Blake's formula

Suppose now that a force vector $\mathbf{f}$ is applied to a viscous fluid in the upper half-space ($x_3 > 0$). Then the corresponding Stokeslet-induced velocity field fails to satisfy the no-slip condition (2). In order to annihilate the velocity field while satisfying the homogeneous Stokes equations in the upper half-space, [4] proposed the following image structure:

$$S_{ij}^W(\mathbf{x},\mathbf{y}) = S_{ij}(\mathbf{x},\mathbf{y}) - S_{ij}(\mathbf{x},\mathbf{y}^I) + 2y_3^2 S_{ij}^D(\mathbf{x},\mathbf{y}^I) - 2y_3 S_{ij}^{SD}(\mathbf{x},\mathbf{y}^I), \tag{10}$$

where $\mathbf{y}^I = (y_1, y_2, -y_3)$ is the reflected image location. Here, S^D is a *modified source doublet* given by

$$S_{ij}^D(\mathbf{x},\mathbf{y}) = \frac{1}{8\pi}(1 - 2\delta_{j3}) \frac{\partial}{\partial x_j} \frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|^3}, \tag{11}$$

and S^{SD} is a *modified Stokes doublet*, given by

$$S_{ij}^{SD}(\mathbf{x},\mathbf{y}) = (1 - 2\delta_{j3}) \frac{\partial S_{i3}(\mathbf{x},\mathbf{y})}{\partial x_j}. \tag{12}$$

Similar, but more involved, decompositions for the Stokes doublet and stresslet in a half-space are given in [6] and [20]. Note that the computation of the modified source doublet S_{ij}^D requires the evaluation of three distinct harmonic dipole fields.

3 A new image formula

In this section, we derive a simpler image structure, using the Papkovich-Neuber representation which involves only a single harmonic function. $\mathbf{y}^I$, as above, will denote the image location

$(y_1, y_2, -y_3)$. Now, however, we define reflected single force, double force, and double force orientation vectors by negating their third components:

$$\mathbf{f}^I = (f_1, f_2, -f_3), \quad \mathbf{g}^I = (g_1, g_2, -g_3), \quad \boldsymbol{\nu}^I = (\nu_1, \nu_2, -\nu_3). \tag{13}$$

We also make use of the harmonic potential due to a unit strength charge,

$$G^S(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}, \tag{14}$$

the harmonic potential due to a unit strength dipole with orientation vector $\boldsymbol{\nu}$,

$$G^D[\boldsymbol{\nu}](\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{3} \nu_i \frac{\partial}{\partial y_i} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}, \tag{15}$$

and the harmonic potential due to a unit strength quadrupole with orientation vectors $\boldsymbol{\nu}$ and $\boldsymbol{\kappa}$,

$$G^Q[\boldsymbol{\nu}, \boldsymbol{\kappa}](\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \nu_i \kappa_j \frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}. \tag{16}$$

Finally, recall that, through the Papkovich-Neuber representation (3), the harmonic function $\phi(\mathbf{x})$ induces velocity and pressure fields that can be written in component form as

$$u_i(\mathbf{x}) = x_3 \frac{\partial}{\partial x_i} \phi(\mathbf{x}) - \delta_{i3}\phi(\mathbf{x}), \quad p(\mathbf{x}) = 2\frac{\partial}{\partial x_3} \phi(\mathbf{x})\,. \tag{17}$$

Of particular note is the fact that at $x_3 = 0$, the only non-zero velocity component is

$$u_3(\mathbf{x}) = -\phi(\mathbf{x}). \tag{18}$$

This suggests a simple two-step strategy. First, annihilate the tangential components of the velocity field induced by a Stokeslet (single layer) or stresslet (double layer) kernel. It is easy to see that this can be accomplished by subtracting the influence of a reflected single force $\mathbf{f}^I = (f_1, f_2, -f_3)$, or double force $\mathbf{g}^I = (g_1, g_2, -g_3), \boldsymbol{\nu}^I = (\nu_1, \nu_2, -\nu_3)$ located at the image point $\mathbf{y}^I$, respectively. It remains only to match the remaining non-zero normal component u_3, which we will do by a judicious choice of the Papkovich-Neuber potential $\phi(\mathbf{x})$.

3.1 The Stokeslet correction

Following the discussion above, let us write out in more detail the velocity field $\mathbf{v}$ remaining after subtracting the image Stokeslet at $\mathbf{y}^I$ from the original Stokeslet at $\mathbf{y}$ (Fig. 1):

$$v_i(\mathbf{x}) = \sum_{j=1}^{3} S_{ij}(\mathbf{x}, \mathbf{y}) f_j - \sum_{j=1}^{3} S_{ij}(\mathbf{x}, \mathbf{y}^I) f_j^I. \tag{19}$$

At the interface $x_3 = 0$, a straightforward computation yields

$$v_1(\mathbf{x}) = 0, \quad v_2(\mathbf{x}) = 0, \quad v_3(\mathbf{x}) = -\frac{f_3^I}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}^I|} - \frac{y_3}{4\pi} \sum_{j=1}^{3} \frac{x_j - y_j^I}{|\mathbf{x} - \mathbf{y}^I|^3} f_j^I. \tag{20}$$

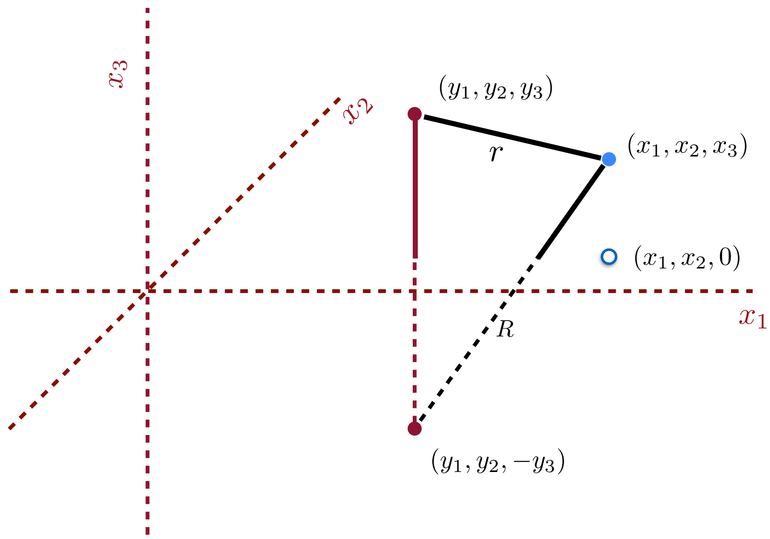

Figure 1: The original source location is in the upper half-space at (y_1, y_2, y_3) and the reflected image source is at $(y_1, y_2, -y_3)$. For target points $(x_1, x_2, 0)$ that lie on the half-space boundary, the distances r and R from the source and image are the same, simplifying the computation of $\mathbf{v}$ in (20).

By inspection of (20), it is clear that the harmonic potential ϕ required to cancel the non-zero normal component v_3 at the interface is that induced by a charge of strength f_3^I and a dipole with orientation vector $\mathbf{f}^I$ of strength y_3 located at $\mathbf{y}^I$. Thus, the velocity field $\mathbf{u}^W$ satisfying the desired no-slip boundary condition (2) can be expressed as:

$$u_i^W(\mathbf{x}) = u_i^A(\mathbf{x}) - u_i^B(\mathbf{x}) - u_i^C(\mathbf{x}), \tag{21}$$

where $\mathbf{u}^A$ and $\mathbf{u}^B$ are the velocity fields induced by the original free-space Stokeslet and the reflected image force vector, respectively:

$$u_i^A(\mathbf{x}) = \sum_{j=1}^{3} S_{ij}(\mathbf{x}, \mathbf{y}) f_j, \qquad u_i^B(\mathbf{x}) = \sum_{j=1}^{3} S_{ij}(\mathbf{x}, \mathbf{y}^I) f_j^I. \tag{22}$$

Here, u_i^C is the Papkovich-Neuber correction

$$u_i^C(\mathbf{x}) = x_3 \frac{\partial}{\partial x_i} \phi(\mathbf{x}) - \delta_{i3} \phi(\mathbf{x}), \tag{23}$$

where the harmonic potential ϕ is that due to a simple charge and dipole, both located at $\mathbf{y}^I$:

$$\phi(\mathbf{x}) = f_3^I \, G^S(\mathbf{x}, \mathbf{y}^I) + y_3 \, G^D[\mathbf{f}^I](\mathbf{x}, \mathbf{y}^I). \tag{24}$$

3.2 The stresslet correction

We state the image structure for the stresslet (the double layer kernel) in the form of a theorem.

Theorem 1. *Let a double force* $\mathbf{g}$ *with orientation vector* $\boldsymbol{\nu}$ *be located at* $\boldsymbol{y}$*, resulting in the free-space velocity field*

$$u_i^T(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}(\boldsymbol{x},\boldsymbol{y})\nu_k g_j \,. \tag{25}$$

Then the corresponding velocity satisfying the no-slip boundary condition (2) is given by

$$u_i^{T,W}(\boldsymbol{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}(\boldsymbol{x},\boldsymbol{y})\nu_k g_j - \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}(\boldsymbol{x},\boldsymbol{y}^I)\nu_k^I g_j^I - \left[x_3\frac{\partial}{\partial x_i}\phi^T(\boldsymbol{x}) - \delta_{i3}\phi^T(\boldsymbol{x})\right], \tag{26}$$

where

$$\phi^T(\boldsymbol{x}) = 2(\boldsymbol{\nu}^I\cdot\mathbf{g}^I)G^D[\mathbf{h}](\boldsymbol{x},\boldsymbol{y}^I) + 2y_3 G^Q[\boldsymbol{\nu}^I,\mathbf{g}^I](\boldsymbol{x},\boldsymbol{y}^I), \tag{27}$$

with $\mathbf{h} = (0,0,1)$*.*

Proof. Note first that the symmetric part of the Stokes doublet is given by

$$T_{ijk}^S(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\left[-\frac{(x_i-y_i)}{|\mathbf{x}-\mathbf{y}|^3}\delta_{jk} + \frac{3(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^5}\right]. \tag{28}$$

Furthermore, the tangential components of the velocity field

$$v_i(\mathbf{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}^S(\mathbf{x},\mathbf{y})\nu_k g_j - \sum_{j=1}^{3}\sum_{k=1}^{3} T_{ijk}^S(\mathbf{x},\mathbf{y}^I)\nu_k^I g_j^I \tag{29}$$

are zero when $x_3 = 0$. Thus, the harmonic potential ϕ^S annihilating v_3 at the interface $x_3 = 0$ is that due to a single quadrupole source located at $\mathbf{y}^I$ with orientation vectors $\boldsymbol{\nu}^I, \mathbf{g}^I$ and strength $2y_3$:

$$\phi^S(\mathbf{x}) = 2y_3 G^Q[\boldsymbol{\nu}^I,\mathbf{g}^I](\mathbf{x},\mathbf{y}^I). \tag{30}$$

It is straightforward to check that the stresslet is simply related to the symmetric part of the Stokes doublet

$$T_{ijk}(\mathbf{x},\mathbf{y}) = \frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^5} = \frac{1}{4\pi}\frac{x_i-y_i}{|\mathbf{x}-\mathbf{y}|^3}\delta_{jk} + T_{ijk}^S(\mathbf{x},\mathbf{y}). \tag{31}$$

Thus, it remains only to annihilate the velocity field induced by the first term on the right-hand side of (31) and its reflected image, given by

$$v_i(\mathbf{x}) = \frac{1}{4\pi}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{x_i-y_i}{|\mathbf{x}-\mathbf{y}|^3}\delta_{jk}\nu_k g_j - \frac{1}{4\pi}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{x_i-y_i^I}{|\mathbf{x}-\mathbf{y}^I|^3}\delta_{jk}\nu_k^I g_j^I \,. \tag{32}$$

At $x_3 = 0$, the tangential components vanish and the normal component is easily computed to be $2(\boldsymbol{\nu}^I\cdot\mathbf{g}^I)G^D[\mathbf{h}](\mathbf{x},\mathbf{y}^I)$, where $\mathbf{h} = (0,0,1)$. The desired result follows. □

3.3 The rotlet correction

Similar representations are easily derived for other fundamental solutions, such as the *rotlet* - the antisymmetric part of the Stokes doublet:

$$T^R_{ijk}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\left[-\frac{(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^3}\delta_{ik} + \frac{(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^3}\delta_{ij}\right]. \tag{33}$$

The corresponding velocity satisfying the no-slip boundary condition (2) is given by

$$u_i^{R,W}(\mathbf{x}) = \sum_{j=1}^{3}\sum_{k=1}^{3} T^R_{ijk}(\mathbf{x},\mathbf{y})\nu_k g_j - \sum_{j=1}^{3}\sum_{k=1}^{3} T^R_{ijk}(\mathbf{x},\mathbf{y}^I)\nu_k^I g_j^I - \left[x_3\frac{\partial}{\partial x_i}\phi^R(\mathbf{x}) - \delta_{i3}\phi^R(\mathbf{x})\right], \tag{34}$$

where the Papkovich-Neuber correction ϕ^R is due to two dipoles:

$$\phi^R(\mathbf{x}) = -2\nu_3^I G^D[\mathbf{g}^I](\mathbf{x},\mathbf{y}^I) + 2g_3^I G^D[\boldsymbol{\nu}^I](\mathbf{x},\mathbf{y}^I). \tag{35}$$

3.4 The Stokes doublet correction

Finally, the Stokes doublet is the sum of its symmetric and antisymmetric parts

$$\begin{aligned} T^D_{ijk}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\Bigg[&-\frac{(x_i-y_i)}{|\mathbf{x}-\mathbf{y}|^3}\delta_{jk} - \frac{(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^3}\delta_{ik} + \frac{(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^3}\delta_{ij} \\ &+ \frac{3(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^5}\Bigg] = T^S_{ijk}(\mathbf{x},\mathbf{y}) + T^R_{ijk}(\mathbf{x},\mathbf{y}). \end{aligned} \tag{36}$$

By combining (30) and (35), we obtain the image formula for the Stokes doublet

$$\begin{aligned} u_i^{D,W}(\mathbf{x}) = &\sum_{j=1}^{3}\sum_{k=1}^{3} T^D_{ijk}(\mathbf{x},\mathbf{y})\nu_k g_j - \sum_{j=1}^{3}\sum_{k=1}^{3} T^D_{ijk}(\mathbf{x},\mathbf{y}^I)\nu_k^I g_j^I \\ &- \left[x_3\frac{\partial}{\partial x_i}\phi^D(\mathbf{x}) - \delta_{i3}\phi^D(\mathbf{x})\right], \end{aligned} \tag{37}$$

where the Papkovich-Neuber potential is given by

$$\phi^D(\mathbf{x}) = -2\nu_3^I G^D[\mathbf{g}^I](\mathbf{x},\mathbf{y}^I) + 2g_3^I G^D[\boldsymbol{\nu}^I](\mathbf{x},\mathbf{y}^I) + 2y_3 G^Q[\boldsymbol{\nu}^I,\mathbf{g}^I](\mathbf{x},\mathbf{y}^I). \tag{38}$$

4 Conclusions

We have derived very simple image formulas for Stokes flow in a half-space induced by any of the standard fundamental solutions - the Stokeslet, stresslet, rotlet, and Stokes doublet. In each case, all that is required is a reflected fundamental solution and a Papkovich-Neuber correction based on a single harmonic potential.

The velocity (and pressure) due to the "direct" and reflected fundamental solutions can be computed together with any software that handles Stokeslets, stresslets, etc. in free space. Furthermore, the Papkovich-Neuber potential requires only the evaluation of a single additional harmonic function — itself requiring only software for free space *harmonic* sources, dipoles and quadrupoles. Many efficient schemes exist for these various steps, such as those described in [10, 11, 28, 27, 9, 25, 13, 26, 24, 22, 8, 12].

A Extension to the two-dimensional problems

The two-dimensional Stokes flow representations in a half-space can be derived similarly. They lead to identical Papkovich-Neuber corrections with corresponding charge, dipole, and quadrupole potentials replaced by their two-dimensional equivalents. Similar to the three-dimensional case, for any $\mathbf{x} \in \mathbb{R}^2$, the velocity field $\mathbf{u}^W$ satisfying the no-slip boundary condition in a half-plane is composed of three terms:

$$\mathbf{u}^W(\mathbf{x}) = \mathbf{u}^A(\mathbf{x}) - \mathbf{u}^B(\mathbf{x}) - \mathbf{u}^C(\mathbf{x}), \tag{39}$$

where the first term is the velocity field induced by the free-space Green's function, the second term is the reflected image about the plane wall annihilating the tangential velocity component, and the third term is a Papkovich-Neuber correction term in the following form:

$$\mathbf{u}^C(\mathbf{x}) = x_2 \nabla \phi(\mathbf{x}) - \begin{bmatrix} 0 \\ \phi(\mathbf{x}) \end{bmatrix}, \quad p(\mathbf{x}) = 2\frac{\partial \phi(\mathbf{x})}{\partial x_2}. \tag{40}$$

The correction potentials for various fundamental solutions are

$$\phi(\mathbf{x}) = f_2^I\, G^S(\mathbf{x}, \mathbf{y}^I) + y_2\, G^D[\mathbf{f}^I](\mathbf{x}, \mathbf{y}^I), \tag{41}$$

for the two-dimensional Stokeslet, and

$$\phi^T(\mathbf{x}) = 2(\boldsymbol{\nu}^I \cdot \mathbf{g}^I) G^D[\mathbf{h}](\mathbf{x}, \mathbf{y}^I) + 2y_2 G^Q[\boldsymbol{\nu}^I, \mathbf{g}^I](\mathbf{x}, \mathbf{y}^I), \tag{42}$$

$$\phi^R(\mathbf{x}) = -2\nu_2^I G^D[\mathbf{g}^I](\mathbf{x}, \mathbf{y}^I) + 2\mathbf{g}_2^I G^D[\boldsymbol{\nu}^I](\mathbf{x}, \mathbf{y}^I), \tag{43}$$

$$\phi^D(\mathbf{x}) = -2\nu_2^I G^D[\mathbf{g}^I](\mathbf{x}, \mathbf{y}^I) + 2g_2^I G^D[\boldsymbol{\nu}^I](\mathbf{x}, \mathbf{y}^I) + 2y_2 G^Q[\boldsymbol{\nu}^I, \mathbf{g}^I](\mathbf{x}, \mathbf{y}^I), \tag{44}$$

for the two-dimensional stresslet, rotlet, and Stokes doublet, respectively, where the images of the source, single force, double force, and double force orientation vectors are

$$\mathbf{y}^I = (y_1, -y_2), \quad \mathbf{f}^I = (f_1, -f_2), \quad \mathbf{g}^I = (g_1, -g_2), \quad \boldsymbol{\nu}^I = (\nu_1, -\nu_2), \tag{45}$$

with the orientation vector $\mathbf{h} = (0, 1)$, and the free-space Laplace Green's functions in two dimensions are given by

$$G^S(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{y}|, \quad G^D[\boldsymbol{\nu}](\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{2} \nu_i \frac{\partial}{\partial y_i} G^S(\mathbf{x}, \mathbf{y}), \tag{46}$$

$$\text{and} \quad G^Q[\boldsymbol{\nu}, \boldsymbol{\kappa}](\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \kappa_j \frac{\partial^2}{\partial y_i \partial y_j} G^S(\mathbf{x}, \mathbf{y}). \tag{47}$$

B Extension to linear elasticity kernels

The single layer kernel for linear isotropic elasticity in R^3 is given by Kelvin's solution [17]

$$U_{ij}(\mathbf{x}, \mathbf{y}) = \frac{1}{8\pi\mu} \left[(2 - \alpha) \frac{\delta_{ij}}{|\mathbf{x} - \mathbf{y}|} + \alpha \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x} - \mathbf{y}|^3} \right], \quad i, j = 1, 2, 3, \tag{48}$$

where λ, μ are Lame's parameters and $\alpha = (\lambda+\mu)/(\lambda+2\mu)$. It is easy to see that the displacement field

$$u_i(\mathbf{x}) = \sum_{j=1}^{3} S_{ij}(\mathbf{x}, \mathbf{y}) f_j - \sum_{j=1}^{3} S_{ij}(\mathbf{x}, \mathbf{y}^I) f_j^I - u_i^C(\mathbf{x}) \tag{49}$$

satisfies the no-displacement boundary condition $\mathbf{u}(\mathbf{x}) = 0$ at $x_3 = 0$, where the Papkovich-Neuber correction is

$$u_i^C(\mathbf{x}) = \alpha x_3 \frac{\partial}{\partial x_i} \phi(\mathbf{x}) - (2-\alpha)\delta_{i3}\phi(\mathbf{x}), \tag{50}$$

$$\phi(\mathbf{x}) = \frac{f_3^I}{\mu} G^S(\mathbf{x}, \mathbf{y}^I) + \frac{y_3}{\mu} G^D[\mathbf{f}^I](\mathbf{x}, \mathbf{y}^I). \tag{51}$$

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